RESEARCH NOTE

Fundamental sloshing frequencies of stratified two-fluid systems in closed prismatic tanks

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The solution to the irrotational eigenvalue problem for internal waves in closed tanks is simple but not easily accessible. Solutions are given for two geometries frequently encountered in industry. It is recommended that acoustic rather than structural elements be used in finite-element calculations of complex geometries

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Surface wave motions of a liquid lying in a vessel of finite dimensions have been of interest in a wide range of contexts, such as tides and harbours[1,2], response of fluids to earthquakes[3], and sloshing of fluids in the fuel tanks of road vehicles, ships and aircraft[4]. Concomitant resonance frequencies are associated with wave patterns the shapes of which depend on the vessel geometry, and classical theory has long been applied to constant-depth vessels with simple cross-sectional areas such as rectangles or circles[1,2]. More complex geometries have required the implementation of numerical codes, usually of the finite-element type.

A more complicated situation which arises in nuclear reactors, and probably in numerous other situations in industry, concerns the interfacial waves in a two-fluid system filling a closed tank. For example, if the two fluids exist at different temperatures, oscillations of the interface could be of concern in relation to thermal stressing of metal components. A literature search performed recently on a computerised data bank failed to locate any relevant material, although Bauer[7] has very recently published calculations of forced oscillations in rectangular containers. Surprisingly, the published material is confined, in the main, to oceanography and is not easily accessible to most engineers.

Two useful references are Thorpe[5] and Roberts[6]. Oceanographers have considered a broad class of density stratifications, and they use the term 'internal seiche', found in lakes and fjords, to describe the discontinuous type addressed here. Most of that material deals with a free top surface, but Thorpe[5] gives details of both experiment and theory with a fixed top surface. Emphasis has been placed on small density differences, associated with salt concentrations, and two-dimensional motion.

In this note, the classical linear approach is applied to a two-fluid system filling a closed, prismatic (in the vertical sense) tank. The underlying mathematics for potential flow is straightforward, and the readers familiar with this aspect of fluid dynamics will know that after separation of variables the problem will degenerate to a dispersion relation coupled to an eigenvalue problem for the two-dimensional Helmholz equation. The dispersion relation has been quoted previously in the context of progressive gravity waves[1,2]. Nevertheless, it seems worthwhile to record the results for two shapes encountered frequently in industry, namely tanks with circular and concentric annuli as cross-sections. The two fluids may be different species, a liquid and its vapour, or even a single fluid sustained at different temperatures, but no account has been taken of the interface thickness so the theory will only apply to waves with heights which are large on the scale of the interface thickness.

General theory

Referring to Fig 1, assumption of irrotational disturbances leads to a convenient representation in terms of the velocity potential Φ such that the fluid velocity $\vec{u}$ is given by grad Φ. The continuity equation for incompressible flow in turn implies that:

$$\nabla^2\Phi = 0 \quad (1)$$

The free modes are analysed by considering time-harmonic motion of radial frequency ω, so that:

$$\Phi(x,y,z,t) = \tilde{\phi}(x,y,z)e^{-i\omega t} \quad (2)$$

viz:

$$\nabla^2\tilde{\phi}_1 = 0 = \nabla^2\tilde{\phi}_2 \quad (3)$$

The boundary conditions at a solid surface are:

$$\vec{n}.\nabla\tilde{\phi} = 0 \quad (4)$$

where $\vec{n}$ is a unit normal to the surface. At the mean

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interface level, combination of the conditions stipulating equality of displacement and pressure, ignoring surface tension, leads to[1,2]:

$$\omega^2\tilde{\phi}_1 - g(1-\rho)\frac{\partial\tilde{\phi}_1}{\partial z} = \rho\omega^2\tilde{\phi}_2 \qquad \rho=\rho_2/\rho_1 \tag{5}$$

Separating variables:

$$\tilde{\phi}_j = \phi_j(x,y)G_j(z) \qquad j=1,2 \tag{6}$$

and one is led to the Helmholz equations:

$$\nabla_\perp^2\phi_j + k^2\phi_j = 0 \tag{7}$$

where $\nabla_\perp^2$ is the Laplacian operator in the x–y plane, and to the classical dispersion relation[1,2] for progressive two-dimensional waves in a stratified two-fluid system. It is convenient to non-dimensionalise lengths with respect to a reference length L:

$$K=kL \qquad H_j=h_j/L \qquad \Omega^2=w^2L/g \tag{8}$$

Then:

$$\Omega^2[\coth(KH_1)+\rho\coth(KH_2)]=(1-\rho)K \tag{9}$$

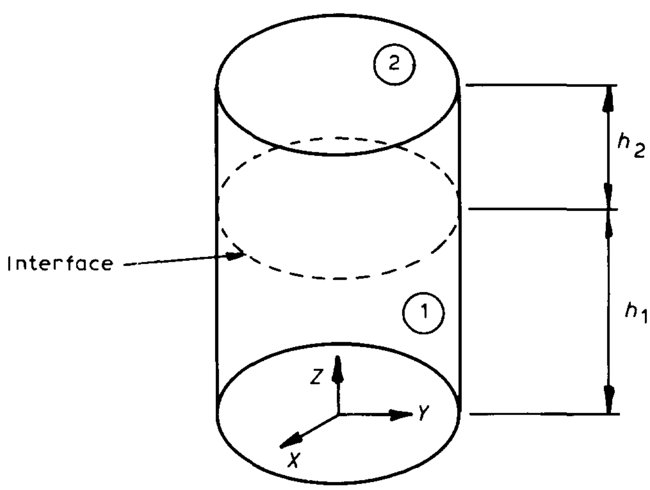

Fig 1 The configuration. The general theory applies to arbitrary cross-sectional shapes

and

$$\nabla_\perp^2\phi + K^2\phi = 0 \qquad \frac{\partial\phi}{\partial n}=0 \tag{10}$$

where all lengths in the Laplacian are now dimensionless.

Application to circles and annuli

In cylindrical coordinates, Eq (10) reads:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + K^2\phi = 0 \tag{11}$$

where r is understood to be dimensionless. Separating variables:

$$\phi = M(r)N(\theta) \tag{12a}$$

$$N_{\theta\theta} + \mu^2 N = 0 \qquad \mu \text{ integer} \tag{12b}$$

$$R^2M_{RR} + RM_R + (R^2-\mu^2)M = 0 \qquad R=Kr \tag{12c}$$

The two independent solutions of Eq (12c) are[8] the Bessel functions $J_\mu(R)$ and $Y_\mu(R)$.

In the case of a circle, only J applies, and if L is chosen to be the radius of the circle, the wavenumbers are given by:

$$J'_\mu(K)=0 \tag{13}$$

For each μ there is an infinite set of solutions to Eq (13), identified by the pair of symbols μ, η. Fig 2 shows Ω versus H_1 for a cylinder the height of which equals the radius, ie $H=1$, for several modes and with $\rho=0.63$.

Turning now to the concentric annulus, defining η as:

$$\eta = r_2/r_1 \tag{14}$$

The solution of Eq (12c) may be written as:

$$M = J_\mu(R) + \xi Y_\mu(R) \tag{15}$$

where ξ is a constant. Applying the conditions at R_1 and R_2, and choosing $L=r_1$:

$$J'_\mu(K)Y'_\mu(\eta K) - J'_\mu(\eta K)Y'_\mu(K) = 0 \tag{16a}$$

$$\xi = -J'_\mu(K)/Y'_\mu(K) \tag{16b}$$

Notation

g	Gravitational acceleration
G	Eq (6)
h_j	Height of layer j
H_j	h_j/L
H	$(h_1+h_2)/L$
i	$(-1)^{1/2}$
J_μ	Bessel function of the first kind
k	Wavenumber
K	kL
L	Reference length
M	Eq (12a)
n	Sub-mode of μ^{th} azimuthal model
N	Eq (12a)
r, θ, z	Cylindrical polar coordinates
r_1, r_2	Radii of annulus
R	Kr
t	Time
u	Fluid velocity vector
x, y, z	Cartesian coordinates
Y_μ	Bessel function of the second kind
η	r_2/r_1
μ	Integer identifying an azimuthal mode
ξ	Constant
ρ_j	Density of fluid j
ρ	ρ_2/ρ_1
ϕ	Eq (6)
$\tilde{\phi}$	Eq (2)
Φ	Velocity potential, Eq (2)
ω	Frequency (rad/s)
Ω	$\omega(L/g)^{1/2}$

Subscripts

j	1 or 2
1	Fluid below the interface
2	Fluid above the interface

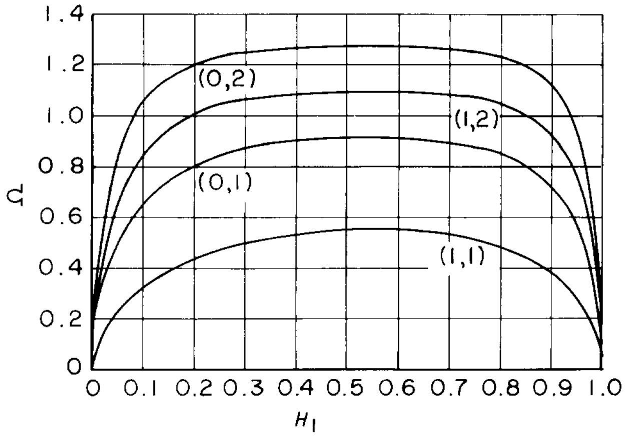

Fig 2 Fundamental frequencies versus interface height for the circle, $\rho=0.63$, H = 1. The pair (μ,n) refers to the nth mode of the azimuthal waves identified by μ

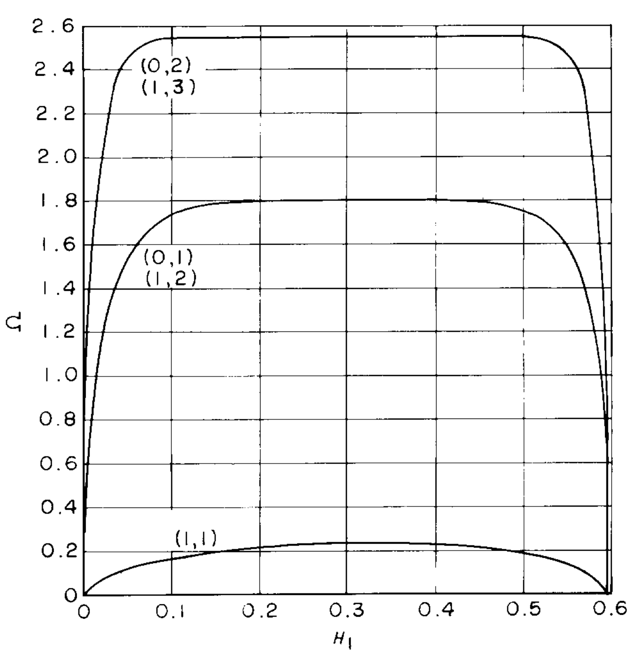

Fig 3 Fundamental frequencies versus interface height for the annulus, $\eta=1.22$, H = 0.595, $\rho=0.63$

As an example, η has been chosen to be 1.22, in which case the first few eigenvalues are:

μ, η	K
0, 1	14.3
0, 2	28.6
1, 1	0.9
1, 2	14.3

Fig 3 gives Ω against H_1 for these modes, with $H=0.595$ and $\rho=0.63$. The first azimuthal mode (1, 1) possesses eigenfrequencies which are much lower than the other modes.

Conclusions

Linear, irrotational theory has been used for determining the resonance frequencies of gravity waves at the interface between two fluids filling a prismatic tank. Semi-analytical solution have been obtained for tanks possessing circles and concentric annuli as cross-sections, and non-dimensional frequencies have been presented as functions of interface height for tanks having specific aspect ratios. More complex cross-sectional geometries have to be solved numerically by a computer code which can determine the eigenvalues for the Helmholz equation (Eq (10)), and since acoustic resonances in rigid cavities are completely analogous, it is recommended that acoustic elements in finite-element codes should be used in preference to structural elements.

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